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EFFECT OF CONVECTION ON THE STABILITY OF A LIQUID WITH A NONUNIFORMLY DISTRIBUTED HEAVY ADMIXTURE

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The effect of the transverse temperature gradient on the stability of steady motion of a viscous incompressible liquid in a plane vertical layer bounded by two infinite solid surfaces is studied. The motion of the liquid is caused by sedimentation of heavy solid spherical particles distributed nonuniformly across the layer and by the horizontal temperature gradient. Spectra of decrements of small normal perturbations are calculated for different particle sizes and different degrees of nonuniformity of the distribution of admixture particles. The stability of a steady flow of the liquid with an admixture decreases with increasing temperature gradient and increasing particle radius and increases with a tendency of the particles to a uniform distribution.

The stability of an isothermal plane–parallel flow of a viscous incompressible liquid (gas) with a small number of heavy solid spherical particles distributed nonuniformly in the flow was studied in [1], where the dependence of flow stability on the character of the distribution of particles in the layer was shown. The effect of the sedimentation rate of the particles, their size, density, heat capacity, and mass concentration on the convective stability of a steady flow of a liquid with a uniformly distributed admixture was considered in [2].

Below we study the effect of the transverse temperature gradient on the stability of a steady motion of a viscous incompressible liquid in a vertical plane layer bounded by two infinite solid plane surfaces. The motion of the liquid is caused by sedimentation of solid spherical particles distributed nonuniformly across the layer and by the temperature difference on the boundaries of the layer.

1. We consider a viscous incompressible liquid containing an admixture of heavy solid particles. The liquid and admixture are assumed to be continuum media penetrating into each other and interacting with each other; the interaction between the particles is ignored. The interaction between the phases during their relative motion obeys the Stokes law. The volume fraction of particles is so small that the Einstein's correction to the liquid viscosity may be ignored. The particles are assumed to be spherical and nondeformable and to have an identical mass m and an identical radius r; the particle density ρ_1 is much greater than the liquid density ρ_0.

First we consider a uniformly heated layer of the liquid. Descending particles move in the liquid in a closed vertical layer between the planes $x = \pm h$. These particles are symmetrically distributed across the layer with respect to the vertical axis z in accordance with the law (see [1])

$$N_0(x,\alpha) = \frac{4\cosh\alpha\cosh(\alpha x/h) - \cosh(2\alpha x/h) - \cosh 2\alpha - 2}{4\cosh\alpha - \cosh 2\alpha - 3},$$

where N_0 is the number of particles per unit volume and α is a coefficient that determines the admixture concentration near the layer boundaries (the solid curves 1 and 2 in Fig. 1 refer to $\alpha = 1$ and 40, respectively).

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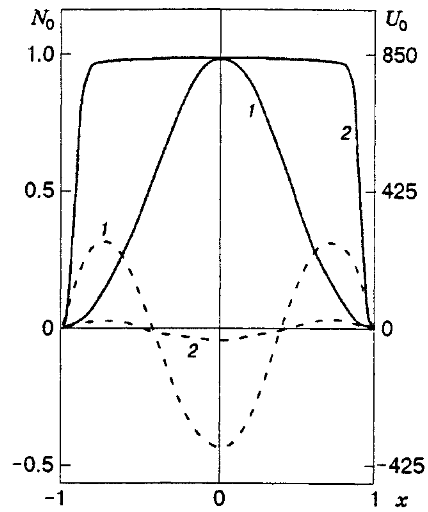

Fig. 1

This formula gives a good description of the distribution of the settling particles in the vertical layer, which is observed in the experiment [3].

Interacting with the liquid, the settling particles distributed nonuniformly across the layer set the liquid into motion. Steady distributions of the liquid and particle velocities in the isothermal case are found from a system of equations that describe the behavior of an incompressible liquid with an admixture of heavy solid particles (see [1]) under the assumption that the trajectories of both liquid and solid particles are straight lines parallel to the vertical axis z and the layer is closed from below and from above at infinity:

$$u_0 = \mathrm{Ga}\, B_1\Big(\frac{4\cosh\alpha\cosh(\alpha x) - \cosh(2\alpha x)/4}{\alpha^2} + B_2x^2 - B_3\Big),$$

$$B_1 = \frac{mh^3}{\rho(4\cosh\alpha - \cosh 2\alpha - 3)}, \quad B_2 = \frac{3}{4\alpha^2}\Big(\frac{15}{4\alpha}\sinh 2\alpha - \frac{7}{2}\cosh 2\alpha - 4\Big), \tag{1.1}$$

$$B_3 = \frac{45}{16\alpha^3}\sinh 2\alpha - \frac{7}{8\alpha^2}\cosh 2\alpha - \frac{1}{\alpha^2}, \quad u_{p0} = u_0 + u_s, \quad u_s = -\mathrm{Ga}\,\tau_v.$$

Dimensionless variables are introduced here; the units of distance, time, velocity, and pressure are h, h^2/ν, ν/h, and $\rho_0\nu^2/h^2$, respectively, $\tau_v = 2r^2\rho_1/(9h^2\rho_0)$ is the dimensionless time during which the particle velocity relative to the liquid decreases by a factor of e as compared to the initial value, $\mathrm{Ga} = gh^3/\nu^2$ is the Galileo number, ν is the kinematic viscosity, and m is the particle mass; the quantities with the subscript p refer to the cloud of particles, u_s is the particle-sedimentation rate, and g is the acceleration of gravity.

Under the action of settling particles, the motion of the liquid is symmetric relative to the channel axis and has two ascending and one descending flows (1.1) [see Fig. 1, where the solid and dashed curves correspond to $N_0(x)$ and $u_0(x)$, respectively]. The intensity of motion rapidly decreases with increasing α (as $\alpha \to \infty$, i.e., with a tendency of the particles to a uniform distribution, we have $u_0 \to 0$).

We study the stability of the exact solution of system (1.1) for the motion of a liquid with nonuniformly distributed heavy particles of the admixture. Small perturbations of a steady plane–parallel flow (1) are assumed to be plane, since the most dangerous disturbances from the viewpoint of origination of instability are two-dimensional ones, as in the case of a pure liquid [4]. We define the stream function of plane perturbations ψ by the relations

$$v_x = -\frac{\partial\psi}{\partial z}, \qquad v_z = \frac{\partial\psi}{\partial x}.$$

Here $\psi(x,z,t) = \varphi(x)\exp[ik(z-ct)]$, $\boldsymbol{u}_p'(x,z,t) = \boldsymbol{v}_p(x)\exp[ik(z-ct)]$ are velocity perturbations of the cloud of particles, k is the real wavenumber, and $c = c_r + ic_i$ is the complex phase velocity of the perturbations.

The equations in dimensionless variables for the amplitudes of perturbations of the stream function φ have the following form:

$$\varphi'''' - 2k^2\varphi'' + k^4\varphi + ik(\varphi'' - k^2\varphi)\Big(c - u_0 - \frac{a_0}{ik\tau_v}\Big) + iku_0''\varphi$$

$$- \frac{a_0}{\tau_v}(v_{px}' - ikv_{px}) + \frac{a_0'}{\tau_v}(v_{pz} - \varphi') - \mathrm{Ga}\, a' = 0, \tag{1.2}$$

$$v_{px} = \frac{ik\varphi}{ik\tau_v(u_{p0} - c) + 1}, \quad v_{pz} = \frac{-\varphi' + u_{p0}'\tau_v v_{px}}{ik\tau_v(u_{p0} - c) + 1}, \quad n = -\frac{ikv_{pz}N_0 + N_0'v_{px} + N_0 v_{px}'}{ik(u_{p0} - c)}.$$

Here $a_0 = N_0 m/\rho_0$; the prime denotes differentiation with respect to the x coordinate.

The condition of adhesion is posed on the solid plane surfaces bounding the liquid layer:

$$\varphi = \varphi' = 0 \quad \text{for} \quad x = \pm 1. \tag{1.3}$$

The boundary-value problem (1.2), (1.3) determines the spectrum of characteristic perturbations and their decrements c_i; the boundaries of stability are found from the condition $c_i = 0$.

In the limiting cases (large viscosity or density of the carrier liquid, small size or density of the particle material), the particle-sedimentation rate may be ignored. Hence, the motion of the liquid does not arise in these cases, and hydrodynamic perturbations decay monotonically (the quiescent state of the liquid with an admixture is stable).

For arbitrary values of the parameters of the problem, the study of the spectrum of small normal perturbations of a steady flow (1.1) and its linear stability reduces to the numerical solution of the spectral amplitude problem (1.2), (1.3). Two linearly independent solutions of Eq. (1.2) can be constructed using the condition $x = -1$ at the left point of the integration domain. Using these particular solutions, we can construct the general solution, which satisfies the boundary conditions for velocity perturbations for $x = 1$. From this we obtain a system of two homogeneous algebraic equations for the coefficients of the general solution. The conditions of existence of a nontrivial solution of this system defines the characteristic equation, which determines the spectrum of the complex eigenvalues of c. The characteristic equation is solved by Newton's iterative method (the relative error was chosen to be 0.001).

For direct numerical integration, Eq. (1.2) was written in the form of a system of four ordinary differential equations of the first order. Their step-by-step integration was performed by the Runge–Kutta–Merson, which allows integration with an automatic choice of the step under a controlled accuracy (the error was 0.01% of the greatest particular solution at this step). The numerical solution involved difficulties caused by the presence of the small parameter $\mathrm{Ga}^{-1} \sim 10^{-4}$ at the highest derivative. Rapidly increasing oscillating solutions appeared. The boundary conditions (1.3) provide the linear independence of particular solutions of (1.2) only at the initial section of integration. Afterwards, because of the rapidly increasing solution and rounding errors, the linear independence of particular solutions is lost: they become close irrespective of the conditions at the initial boundary of numerical integration. For this reason, the problem to be solved is ill-posed, and the characteristic decrements cannot be determined. To recover the linear independence of particular solutions, the method of orthogonalization [5] was used. At each step of integration, the vector-solutions were orthonormalized to the vector-solution that had the maximum absolute value (at this step) using the Gram–Schmidt procedure. Linear transformations used in orthogonalization do not alter the eigenvalues of the spectral problem.

The presence of admixture particles affects, first of all, the spectrum of perturbation decrements. In contrast to the spectrum of the pure liquid [4], the spectrum of perturbations here is considerably wider due to the appearance of perturbations related to the cloud of particles. Vibrational perturbations appear in the layer (see [1]). It should be noted that the stability of the liquid layer flow with nonuniformly distributed heavy solid particles of an admixture is caused by interaction of counterflows: the descending central flow and two ascending flows near the walls. The instability of motion is caused by the lower modes of hydrodynamic

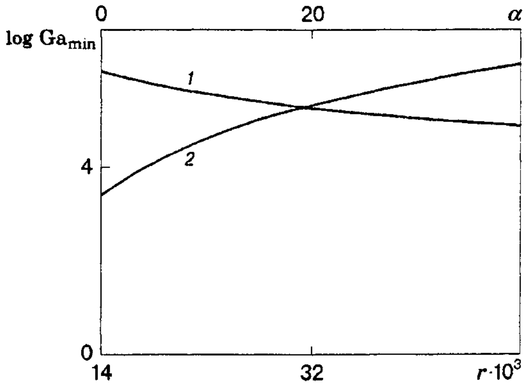

Fig. 2

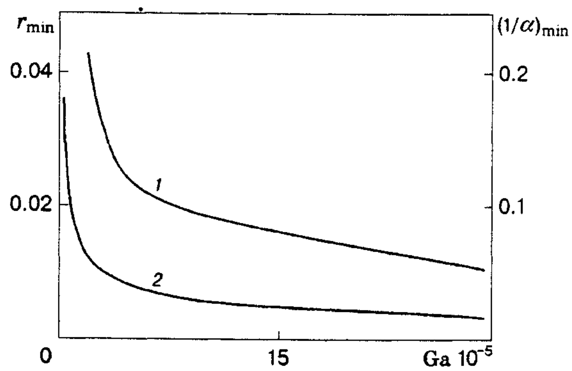

Fig. 3

perturbations, and the decrements of normal perturbations are complex. The settling particles give birth to vibrational (travelling) perturbations.

The calculations show that the flow intensity increases and its stability decreases with increasing particle radius r. Curve 1 in Fig. 2 shows the dependence $\log \mathrm{Ga}_{\min}$ ($\mathrm{Ga}_{\min}$ is the minimum critical Galileo number above which the motion is unsteady) on the particle radius r for $\alpha = 10$ and $\rho_1/\rho_0 = 500$ (this ratio of ρ_1/ρ_0 corresponds to sedimentation of sawdust in air). With increasing parameter α, the flow stability increases, since its intensity decreases (curve 2 in Fig. 2, $r = 0.05$). Figure 3 shows the minimum critical radius of the particles $r_{\min}$ and $(1/\alpha)_{\min}$ on Ga (curves 1 and 2, respectively). For $r > r_{\min}$, the motion becomes unsteady. The characteristic of the isothermal flow $(1/\alpha)_{\min}$ determines its intensity. The parameter $1/\alpha$ determines the influence of particle distribution across the vertical layer on the stability of the isothermal motion of the liquid, which is caused by nonuniformly distributed settling particles. An increase in the parameter $1/\alpha$ corresponds to an increase in the velocity of steady motion of the liquid due to the increase in inhomogeneity of distribution of particles, which concentrate in the middle of the layer (solid curve 1 in Fig. 1). Thus, we can affect the flow stability without changing the parameters of the particles and carrier liquid only by changing the concentration of admixture particles across the layer.

It follows from Fig. 3 that the stability decreases with decreasing viscosity (increasing Ga). The phase velocity of hydrodynamic perturbations c_r decreases with increasing Ga ($\alpha = \mathrm{const}$) and increases for $r = \mathrm{const}$, which agrees with the results reported in [6].

2. We consider the motion of a nonuniformly heated liquid with an admixture. The equations of free convection of an incompressible liquid with an admixture of heavy solid particles, which develops on the background of a steady isothermal flow, have the following form in the Boussinesq approximation (see [4]) when written in the dimensionless form:

$$\frac{\partial \boldsymbol{u}}{\partial t} + ((\boldsymbol{u}_0 + \boldsymbol{u})\nabla)(\boldsymbol{u}_0 + \boldsymbol{u}) = -\nabla p + \Delta \boldsymbol{u} + \frac{a_0(\boldsymbol{u}_p - \boldsymbol{u})}{\tau_v} + (1 + a_0)\,\mathrm{Gr}\,T\boldsymbol{\gamma},$$

$$\frac{\partial \boldsymbol{u}_p}{\partial t} + ((\boldsymbol{u}_{p0} + \boldsymbol{u}_p)\nabla)(\boldsymbol{u}_{p0} + \boldsymbol{u}_p) = -\frac{\boldsymbol{u}_p - \boldsymbol{u}}{\tau_v},$$

$$\frac{\partial T}{\partial t} + (\boldsymbol{u}_0 + \boldsymbol{u})\nabla T = \frac{\Delta T}{\mathrm{Pr}} + \frac{a_0 b(T_p - T)}{\tau_t}, \quad \frac{\partial T_p}{\partial t} + (\boldsymbol{u}_{p0} + \boldsymbol{u}_p)\nabla T_p = -\frac{T_p - T}{\tau_t}, \tag{2.1}$$

$$\operatorname{div} \boldsymbol{u} = 0, \qquad \frac{\partial N}{\partial t} + \frac{\nabla(N(\boldsymbol{u}_{p0} + \boldsymbol{u}_p) + N_0 \boldsymbol{u}_p)}{\mathrm{Pr}} = 0,$$

$$\tau_t = 3\,\mathrm{Pr}\,\tau_v b/2, \quad b = C_1/C, \quad \mathrm{Pr} = \nu/\chi, \quad \mathrm{Gr} = g\beta\Theta h^3/\nu^2.$$

Here $\boldsymbol{u}$ is the velocity of a convective flow arising on the background of steady isothermal motion with a velocity $\boldsymbol{u}_0$, T is the temperature, p is the pressure of the liquid counted from the overnormalized hydrostatic

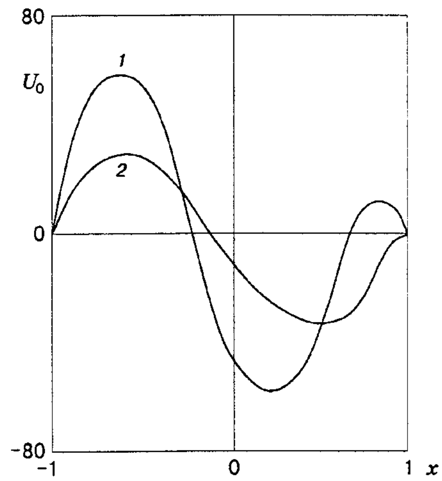

Fig. 4

pressure due to the presence of settling particles, C is the heat capacity of the liquid at constant pressure, β and χ are the coefficient of volume expansion of the liquid and its temperature diffusivity, C_1 is the heat capacity of the particle material, the characteristic temperature is the half-difference of temperatures at the boundaries of the layer Θ, τ_t is the dimensionless time needed for the temperature difference between the liquid and particles to decrease by a factor of e as compared to the initial value, Pr and Gr are the Prandtl and Grashof numbers, and $\boldsymbol{\gamma}$ is the unit vector directed vertically upward.

We find the exact steady solution of system (2.1) in the presence of a constant horizontal temperature gradient when a convective flow caused by the nonuniform distribution of the liquid temperature is superimposed on the steady motion of the liquid u_0 caused by interaction with nonuniformly distributed settling solid particles. Using the conditions of adhesion of the liquid to the solid boundaries of the layer and the flow closure, we obtain the expression for the steady velocity of the liquid:

$$U_0 = \mathrm{Gr}\left\{\frac{x^3}{6} + \frac{m}{\rho_0(4\cosh\alpha - \cosh 2\alpha - 3)}\left[\frac{4\cosh\alpha}{\alpha^2}\left(x\cosh(\alpha x) - \frac{2\sinh(\alpha x)}{\alpha}\right)\right.\right.$$

$$\left.\left. - \frac{1}{4\alpha^2}\left(x\cosh(2\alpha x) - \frac{\sinh(2\alpha x)}{\alpha}\right) - \frac{\cosh 2\alpha + 2}{6}x^3\right] + C_1 x\right\} + u_0, \qquad (2.2)$$

$$U_{p0} = U_0 + u_s, \qquad T_0 = T_{p0} = -x,$$

where

$$C_1 = -\frac{1}{6} - \frac{m}{2\rho_0(4\cosh\alpha - \cosh 2\alpha - 3)}$$

$$\times\left[\frac{8\cosh\alpha}{\alpha^2}\left(\cosh\alpha - \frac{2\sinh 2\alpha}{\alpha}\right) - \frac{1}{2\alpha^2}\left(\cosh 2\alpha - \frac{\sinh 2\alpha}{\alpha}\right) - \frac{\cosh 2\alpha + 2}{3}\right].$$

Under the action of settling particles and the horizontal temperature gradient, an asymmetric flow of the liquid with two ascending and one descending flows is formed in the layer. The intensity of motion decreases with increasing α. Curves 1 and 2 in Fig. 4 correspond to $\alpha = 1$ and 40, respectively. In the limiting case of uniformly distributed particles ($\alpha \to \infty$, i.e., $N_0 = \mathrm{const}$ and $a_0 = \mathrm{const}$), we obtain from formulas (2.2) that the isothermal component of velocity is $u_0 = 0$, and the convective component is a usual cubic profile (see [2, 4]).

3. To study the stability of motion of a nonuniformly heated medium containing nonuniformly distributed settling particles, we consider the disturbed fields of velocity, temperature, pressure, and the number of particles per unit volume: $\boldsymbol{U}_0 + \boldsymbol{v}$, $\boldsymbol{U}_{p0} + \boldsymbol{v}_p$, $T_0 + T$, $T_{p0} + T_p$, $p_0 + p$, and $N_0 + N$, where $\boldsymbol{v}$, $\boldsymbol{v}_p$, T, T_p, p, and N are small perturbations. The equations for perturbations can be derived from (1.1) using linearization over the perturbations.

As in the cases of a pure liquid [7] and a liquid with a uniformly distributed admixture [2, 8], we can show for a mixture with a nonuniformly distributed admixture of heavy solid particles that the problem of stability of convective flow to spatial perturbations reduces to the corresponding problem for plane perturbations. In the case of the vertical orientation of the layer, plane perturbations are more dangerous, i.e., they lead to instability more rapidly than spatial perturbations (with lower values of the critical Galileo and Grashof numbers). Hence, in stability studies, we may confine ourselves to plane perturbations.

We consider plane normal perturbations

$$v_x = -\frac{\partial \psi}{\partial z}, \qquad v_z = \frac{\partial \psi}{\partial x}, \qquad \psi(x,z,t) = \varphi(x)\exp[ik(z-ct)],$$
$$T(x,z,t) = \theta(x)\exp[ik(z-ct)], \qquad N(x,z,t) = n(x)\exp[ik(z-ct)], \tag{3.1}$$

where ψ is the stream function and φ, θ, and n are the perturbation amplitudes.

As a result, we obtain dimensionless equations for perturbation amplitudes from (2.1) in the linear approximation, taking into account (3.1):

$$\varphi'''' - 2k^2\varphi'' + k^4\varphi + ik(\varphi'' - k^2\varphi)\Big(c - U_0 + \frac{a_0}{ik\tau_v}\Big(\frac{1}{A} - 1\Big)\Big)$$
$$+ \frac{a_0'\varphi'}{\tau_v}\Big(\frac{1}{A} - 1\Big) + \varphi\Big(ikU_0'' + \frac{ik}{A^2}(a_0\,U_{p0}'' + a_0'U_{p0}') + \frac{2a_0k^2\tau_v(U_{p0}')^2}{A^3}\Big)$$
$$+ (1 + a_0)\,\mathrm{Gr}\,\theta' + a'\,\mathrm{Gr}\,T + a\,\mathrm{Gr}\,T' + a_0'\,\mathrm{Gr}\,\theta = 0, \tag{3.2}$$
$$\frac{1}{\mathrm{Pr}}(\theta'' - k^2\theta) + \theta\Big(\frac{a_0 b}{\tau_t}\Big(\frac{1}{B} - 1\Big) + ik(c - U_0)\Big) + ik\varphi T'\Big(\frac{a_0 b}{AB} + 1\Big) = 0.$$

Here $A = ik\tau_v(U_{p0} - c) + 1$, $B = ik\tau_t(U_{p0} - c) + 1$, and $n = \tau_v/(A-1)(2N_0k^2\tau_vU_{p0}'\varphi/A^2 + N_0'ik\varphi/A)$.

The boundary conditions have the form

$$\varphi = \varphi' = \theta = 0 \quad \text{for} \quad x = \pm 1. \tag{3.3}$$

The boundary-value problem (3.2), (3.3) determines the spectrum of perturbation decrements and the boundaries of stability ($c_i = 0$) of motion of a nonuniformly heated liquid containing admixture particles distributed nonuniformly across the layer. To solve this boundary-value problem, we also used the Runge–Kutta–Merson method with step-by-step integration with the Gram–Schmidt orthogonalization of solutions at each step of integration.

4. The presence of admixture particles changes the spectra of the decrements of hydrodynamic and thermal perturbations: vibrational perturbations appear, which are related to the cloud of particles. The instability of the flow of a nonuniformly heated layer of a liquid with nonuniformly distributed heavy solid particles of an admixture is caused by interaction of counterflows. The instability of motion is caused by the lower modes of hydrodynamic perturbations, and the decrements of thermal perturbations are negative. Obviously, this is related to the prevalence of the isothermal component in the steady flow examined for stability. The decrements of normal perturbations are complex. The settling particles generate vibrational perturbations and favor their transport.

We consider the results of investigation of the effect of convection on stability of steady motion of the liquid, which is caused by sedimentation of heavy solid particles of the admixture distributed nonuniformly across the vertical layer. For the parameters of the carrier medium and admixture chosen in the present work, the steady convective flow is a small addition to the steady isothermal motion (the maximum velocity of the isothermal flow is more than ten times greater than the maximum velocity of the convective flow). In the case considered, the hydrodynamic modes of the spectra of perturbation decrements ($\mathrm{Pr} = 0.73$) almost

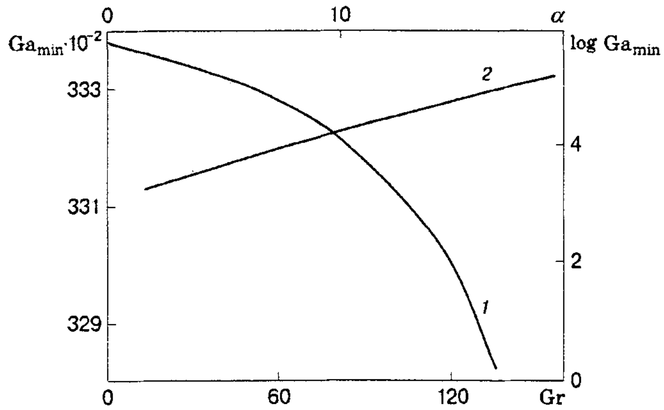

Fig. 5

coincide with the spectra obtained in the isothermal case, and it is hydrodynamic perturbations that disturb the stability. With increasing Gr, the convective velocity increases, and the flow stability decreases. Curve 1 in Fig. 5 shows the dependence of the minimum critical Galileo number on the Grashof number ($\rho_1/\rho_0 = 500$, $\alpha = 10$, Pr = 0.73, $r = 0.05$, and $b = 2.7$, which corresponds to saw-dust in air). With increasing Gr, the critical Galileo number decreases, and the phase velocity of perturbations increases. This is also observed with increasing radius of admixture particles: with increasing particle radius from 0.015 to 0.05, the value of Ga_{min} decreases almost by a factor of 70. With increasing parameter α, the intensity of the isothermal component of the flow decreases, and the flow stability increases (curve 2 in Fig. 5).

A comparison of the data obtained in Sec. 3 with the results reported in [1] and Sec. 1 of the present work shows that the effect of weak convection on the stability of an isothermal flow of a liquid caused by sedimentation of nonuniformly distributed heavy particles is insignificant (the threshold of stability decreases by approximately 5–7%). The stability of a flow of a nonuniformly heated liquid with an admixture increases significantly with increasing parameter α, which characterizes the degree of uniformity of particle distribution across the layer, approaching the stability of a liquid flow with a uniformly distributed admixture (see [2]). The growth in particle sizes leads to a significant decrease in the stability of motion.

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